Morse Homology: A Brief Introduction

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Abstract

After presenting some preliminary results from Morse theory, we discuss topics such as gradient fields, the Smale condition, and the Morse complex. Then we define the Morse homology and showcase its usefulness with concrete examples.

I. Introduction

This paper is being written as part of an independent study for the spring'16 semester at CUNY Hunter College. Our goal is to present a (very) basic introduction to Morse homology and, eventually, to its infinite-dimensional analogue: Floer homology.[1] Before we even dream of tackling the heavy machinery of Morse homology however, we need to present some of the language and basic results that we need in order to develop the theory in the following sections. This is by no means intended to be a complete, rigorous treatment of the deep subject of Morse theory, but rather an attempt to summarize the essential material that is required to define the Morse complex (which is given by the critical points of a Morse function and the trajectories of a gradient field) and compute the associated homology groups. The reader is encouraged to explore the provided references to further expand his/her knowledge of this wonderful subject and all its ramifications.

II. A Sprinkle of Morse Theory

Definition 1. *Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a C^∞ function. A **critical point** of f is a point $x \in M$ such that $\mathrm{d}f_x = 0$; that is, the induced linear map $\mathrm{d}f\colon T_xM \to T_{f(x)}\mathbb{R}$ is zero. The real number $f(x)$ in such case is called a **critical value** of f.*

If x is a critical point of f, we can define a covariant 2-tensor, i.e. a symmetric bilinear form $\mathrm{d}^2 f_x\colon T_xM \times T_xM \to \mathbb{R}$ as follows: If $v, w \in T_xM$, then there are extensions to vector fields V and W such that

$$\mathrm{d}^2 f_x(v, w) = V_x(Wf(x)), \tag{1}$$

where V_x is by definition just v. Now consider a local coordinate chart (x^i) and let

$$v = \sum_i V^i \frac{\partial}{\partial x^i}\Big|_x \qquad \text{and} \qquad w = \sum_j W^j \frac{\partial}{\partial x^j}\Big|_x.$$

[1] Due to time constraints we only discuss Morse homology in this paper. Floer homology and related topics will be discussed in a sequel.

(We can also take $W = \sum_j W^j \, \partial/\partial x^j$, where W^j is now a constant function.) Then,

$$\mathrm{d}^2 f_x(v,w) = V_x(Wf(x)) = v(Wf)(x) = v\left(\sum_j W^j \frac{\partial f}{\partial x^j}\right)(x) = \sum_{i,j} V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j}(x),$$

so that the matrix

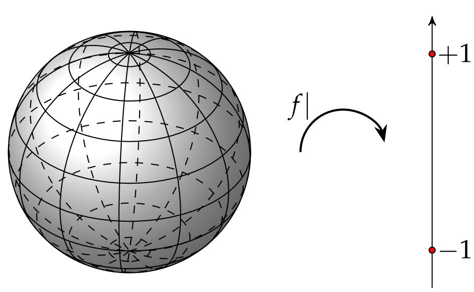
$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(x)\right) \tag{2}$$

represents $\mathrm{d}^2 f_x$ with respect to the basis $\partial/\partial x^1|_x, \ldots, \partial/\partial x^n|_x$. (This matrix is known as the ***Hessian*** of f.)

Definition 2. *We will say that a critical point x is* ***degenerate*** *if the bilinear form $\mathrm{d}^2 f_x$ is degenerate (i.e. if $\mathrm{d}^2 f_x = 0$); we call x* ***nondegenerate*** *otherwise. Moreover, we will say that a function is a* ***Morse function*** *if all its critical points are nondegenerate.*

Remark 1. *It is clear that a critical point x is nondegenrate if and only if the matrix* (2) *is nonsingular (i.e. invertible).*

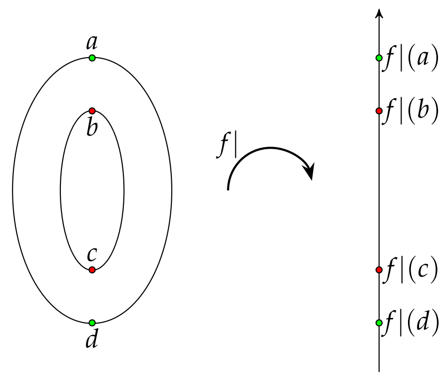
It is a fact that studying a well-chosen function on a manifold can give rather precise information on its topology. A most instructive example is known as the "height" function $f\colon \mathbb{R}^3 \to \mathbb{R}$ and its restriction $f|$ to the different submanifolds embedded (or immersed) in $\mathbb{R}^3$, as presented in the following figures.



In the figure above, note that for $a \in \mathbb{R}$, the level sets of $f|$ are

$$f|^{-1}(a) = \begin{cases} \emptyset & \text{if } a < -1, \\ \text{a point} & \text{if } a = -1, \\ \text{a circle} & \text{if } -1 < a < 1, \\ \text{a point} & \text{if } a = 1, \\ \emptyset & \text{if } a > 1. \end{cases}$$

Each of these level sets has a well defined topology which changes exactly at the regular points of the function; in this case at the north pole (maximum) and the south pole (minimum) of the 2-sphere. The situation is analogous for the torus $\mathbb{T}^2$ in the next figure, except that there are now critical points that are not extrema of the function, namely the two "saddle" points b and c:



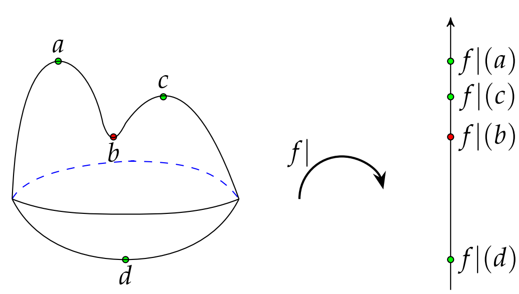
The corresponding level sets at these saddle points b and c are curves of the form $\bigcirc\!\!\!\bullet\!\!\!\bigcirc$ (that is, $\mathbb{S}^1 \vee \mathbb{S}^1$), and are therefore not submanifolds (immersed or embedded). The regular, noncritical level sets must however all be (embedded) submanifolds because of the regular level set theorem.[2]

Let $\mathbb{T}^2_a$ be the set of all points $x \in \mathbb{T}^2$ such that $f|(x) \leq a$. Then we have the following:

$$\mathbb{T}^2_a \begin{cases} = \varnothing & \text{if } a < f|(d), \\ \cong \mathbb{D}^2 & \text{if } f|(d) < a < f|(c), \\ \cong \mathbb{S}^1 \times I \text{ (a cylinder)} & \text{if } f|(c) < a < f|(b), \\ \cong \mathbb{T}^2 \smallsetminus \mathring{\mathbb{D}}^2 & \text{if } f|(b) < a < f|(a), \\ = \mathbb{T}^2 & \text{if } a > f|(a). \end{cases}$$

Remark 2. *In order to describe the changes in the topology of $\mathbb{T}^2_a$ as a passes through the critical values of $f|$, it is convenient to consider homotopy type rather than homeomorphism type (for a somewhat detailed discussion the reader is referred to [MJ]).*

Now let us take a second look at the "height" function on the 2-sphere, although this time we consider a "deformed" sphere, which we denote by $\mathbb{S}^2_*$. Obviously, $\mathbb{S}^2_*$ is still diffeomorphic to $\mathbb{S}^2$, but the function now has two local maxima and a saddle point:



[2] Recall that this theorem states that every regular level set of a smooth map between smooth manifolds (or of a smooth function to $\mathbb{R}$ in our particular case) is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Note that the parity of the number of critical points of the new function is the same as that of the original one (i.e., they are equal mod 2). It turns out that if we have that the critical points of our function are nondegenerate (which is indeed the case for the critical points shown in our figures), then modulo 2 the number of critical points equals the Euler characteristic of the manifold, an invariant that does not depend on the function but only on the manifold itself. However, note that this invariant is not especially strong:

$$\chi(\mathbb{S}^2) = 2, \quad \chi(\mathbb{T}^2) = 0 \quad \Longrightarrow \quad \chi(\mathbb{S}^2) \overset{\text{mod } 2}{=\!=} \chi(\mathbb{T}^2),$$

but it is clear that the sphere and the torus are very different manifolds even if the two admit a function with four nondegenerate critical points (and hence the same Euler characteristic mod 2). It is the concept of Witten spaces of trajectories that will allow us to present a finer invariant (known as the ***Morse homology*** $HM_k(V)$ of a manifold V (c.f. §IV)) in the following sections.

We close out this section by presenting the Morse Lemma, which is, for our purposes, the most important result of Morse theory. We refer the reader to [MJ] for a proof using Hadamard's lemma, or to [AD] for a proof that is a direct application of the implicit function theorem.

Definition 3. *A symmetric bilinear form g on a vector space V is said to be **negative definite** provided that for $v \in V$ and $v \neq 0$, we have $g(v,v) < 0$. The **index** of g on V is the largest integer that is the dimension of a subspace $W \subseteq V$ on which $g|_W$ is negative definite. We will refer to the index of the symmetric bilinear form $\mathrm{d}^2 f_x$ which we previously defined on* (1) *simply as the **index of f at** x.*

The Morse Lemma will show that the behavior of a function f at p can be completely described by its index:

Lemma 1 (Morse Lemma). *Let $p \in M$ be a nondegenerate critical point for $f \in C^\infty(M)$. Then there is a local smooth chart (U, φ), where U is a neighborhood of p and $\varphi(p) = (x^1(p), \dots, x^n(p)) = (0, \dots, 0)$, and such that*

$$f \circ \varphi^{-1}(x^1, \dots, x^n) = f(p) - \sum_{j=1}^{\nu} (x^j)^2 + \sum_{j=\nu+1}^{n} (x^j)^2,$$

where ν is the index of f at p.

Definition 4. *A chart in whose open set the coordinates given by the Morse lemma are defined is called a **Morse chart**.*

Remark 3. *An immediate result that follows from the Morse lemma is the fact that the nondegenerate critical points of a function are isolated. This implies, in particular, that a Morse function on a compact manifold can only have finitely many critical points.*

III. Pseudo-Gradients

Let (M, g) be a Riemannian manifold and recall the notion of the gradient of a function $f \in C^\infty(M)$, denoted $\operatorname{grad} f$, which is given by

$$\operatorname{grad} f = (\mathrm{d}f)^\sharp = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Unraveling the definitions, we see that for any vector field $X \in \Gamma^\infty(TM)$, the gradient satisfies

$$g(\operatorname{grad} f, X) = (\operatorname{grad} f)^\flat(X) = \mathrm{d}f(X) = Xf.$$

(Here we are using the "sharp" ($\sharp$) and "flat" ($\flat$) operators, which are the well known *musical isomorphisms* between TM and T^*M.) Now we have the following definition:

Definition 5. *Let $f\colon M \to \mathbb{R}$ be a Morse function on a manifold M. A* ***pseudo-gradient field*** *(also known as* ***pseudo-gradient adapted to*** *f) is a vector field V on M such that:*

- *We have $\mathrm{d}f_x(V_x) \leq 0$, where equality holds if and only if x is a critical point.*
- *In a Morse chart in the neighborhood of a critical point, V coincides with the negative gradient for the canonical metric on $\mathbb{R}^n$.*

This notion allows us to make the Morse charts more precise by specifying the trajectories (also known as *integral curves* or *flow lines*) of a pseudo-gradient field. For example, Figure 1 shows the difference between a maximum and a minimum critical points in a Morse chart.

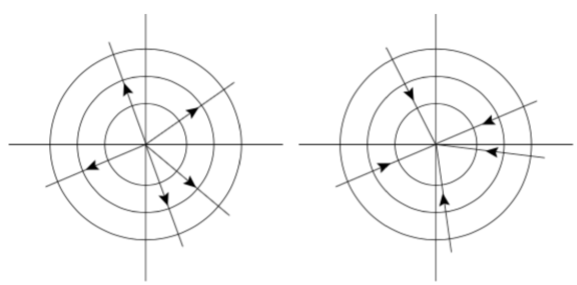


Figure 1: *Trajectories of a pseudo-gradient field in a Morse chart at a maximum (left) and a minimum (right).*

Recall that a ***flow domain*** for a manifold M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\}$ is an open interval containing 0. A ***flow*** on M is then a continuous map $\theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: for all $p \in M$, we have $\theta(0, p) = p$, and and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,

$$\theta(t, \theta(s, p)) = \theta(t + s, p).$$

Now we make the following important definition:

Definition 6. *Let $p \in M$ be a critical point of $f \in C^\infty(M)$. Denote by θ_t the associated map $\theta_t \colon M \to M$ of the flow $\theta \colon \mathcal{D} \to M$ of a pseudo-gradient. We define the* ***stable manifold*** *of p to be*

$$W^s(p) = \left\{ x \in M \mid \lim_{t \to +\infty} \theta_t(x) = p \right\},$$

and its ***unstable manifold***

$$W^u(p) = \left\{ x \in M \mid \lim_{t \to -\infty} \theta_t(x) = p \right\}.$$

Remark 4. *Note that with regards to stable/unstable manifolds, it only makes sense to consider $\mathcal{D} = \mathbb{R} \times M$, i.e. we are only going to focus on global flows.*

Remark 5. *The stable and unstable manifolds of the critical point p are submanifolds of M that are diffeomorphic to open disks. Moreover, we have*

$$\dim W^u(p) = \operatorname{codim} W^s(p) = \operatorname{Ind}(p),$$

where $\operatorname{Ind}(p)$ denotes the index of the point p as a critical point of f.

To illustrate the concept of stable/unstable manifolds, here is a very simple example. Consider the 2-sphere $\mathbb{S}^2$ with the height function f described earlier. Let p be the minimum (south pole) and let q be the maximum (north pole). Then, for any pseudo-gradient field, we have

$$W^s(p) = \mathbb{S}^2 \setminus \{q\}, \qquad W^u(p) = \{p\},$$

and similarly

$$W^s(q) = \{q\}, \qquad W^u(q) = \mathbb{S}^2 \setminus \{p\}.$$

The most important property of the trajectories of a vector field is that they all connect critical points of a function: all trajectories come from a critical point and go towards another critical point:

Proposition 1. *Suppose that M is a compact manifold. Let $\gamma \colon \mathbb{R} \to M$ be a trajectory of the pseudo-gradient field V. Then there exist critical points c and d of $f \in C^\infty(M)$ such that*

$$\lim_{t \to -\infty} \gamma(t) = c \qquad \text{and} \qquad \lim_{t \to +\infty} \gamma(t) = d.$$

Proof. A proof can be found on [AD, 28-29]. □

Now we are finally arriving at the main result of this section. First let us recall that on a manifold M two embedded submanifolds $S, S' \subseteq M$ are said to ***intersect transversely*** if for each $p \in S \cap S'$, the tangent spaces T_pS and T_pS' together span T_pM (where we consider T_pS and T_pS' as subspaces of T_pM). We denote the intersection of two submanifolds S, S' that meet transversally as $S \pitchfork S'$.

Definition 7. *A pseudo-gradient field adapted to the Morse function f is said to satisfy the* ***Smale condition*** *if all stable and unstable manifolds of its critical points intersect transversally, that is, if for all critical points p, q of f, we have*

$$W^u(p) \pitchfork W^s(q).$$

Remark 6. *Certain stable and unstable manifolds always meet transversally. For example, we always have:*

- $W^u(p) \pitchfork W^s(p)$ *(for the same critical point p), which is what we see in a Morse chart around p.*
- $W^u(p) \cap W^s(q) = \emptyset$ *if p and q are distinct and $f(p) \leq f(q)$ (in particular, these stable and unstable manifolds are transversal).*

If the vector field satisfies the Smale condition, then for all critical points p and q, we have

$$\operatorname{codim}(W^u(p) \cap W^s(q)) = \operatorname{codim} W^u(p) + \operatorname{codim} W^s(q);$$

that is,

$$\dim(W^u(p) \cap W^s(q)) = \operatorname{Ind}(p) - \operatorname{Ind}(q).$$

Under our condition, this intersection $W^u(p) \pitchfork W^s(q)$ is a submanifold of M, which we will denote by $\mathcal{M}(p, q)$. It consists of all points on the trajectories connecting p to q:

$$\mathcal{M}(p,q) = \left\{ x \in M \mid \lim_{t \to -\infty} \theta_t(x) = p \quad \text{and} \quad \lim_{t \to +\infty} \theta_t(x) = q \right\}.$$

IV. Morse Homology

Recall that a ***chain complex*** is a sequence C of modules endowed with linear maps $\partial \colon C_k \to C_{k-1}$ that satisfy $\partial \circ \partial = \partial^2 = 0$. Now let C and D be complexes of $\mathbb{Z}_2$-vector spaces. Then their tensor product is the complex defined by

$$(C \otimes D)_k = \otimes_{i+j=k} C_i \otimes D_j$$

with boundary operator

$$\begin{aligned} \partial_k^{C \otimes D}(c \otimes d) &= \Big((\partial_i^C c) \otimes d, c \otimes (\partial_j^D d) \Big) \\ &\in C_{i-1} \otimes D_j \oplus C_i \otimes D_{j-1} \\ &\subset (C \otimes D)_{k-1} \qquad\qquad \text{for } c \otimes d \in C_i \otimes D_j \subset (C \otimes D)_k. \end{aligned}$$

It can be shown then that the homology of the tensor product complex is the tensor product of the homologies; i.e., $H_*(C \otimes D) = H_*(C) \otimes H_*(D)$ (see [AD, Proposition B.11, Pg 556]).

Now, letting $\operatorname{Crit}_k(f)$ denote the set of critical points of index k of a function f, we define the vector space

$$C_k(f) = \left\{ \sum_{p \in \operatorname{Crit}_k(f)} \alpha_p \, p \mid \alpha_p \in \mathbb{Z}_2 \right\}.$$

In other words, for every integer k, we let $C_k(f)$ be the $\mathbb{Z}_2$-vector space generated by the critical points of index k of f (we will often denote this simply by C_k when there is no risk of confusion). Using the connections between critical points established by the trajectories of a generic (that is, satisfying the Smale condition) pseudo-gradient field V, we define maps

$$\partial_V : C_k \to C_{k-1}.$$

In order to define such map ∂_V on $C_k(f)$, it suffices to know how to define $\partial_V(p)$ for a critical point p of index k. This must be a linear combination of the critical points of index $k-1$:

$$\partial_V(p) = \sum_{q \in \mathrm{Crit}_{k-1}} \eta_V(p,q)q \qquad \text{with } \eta_V(p,q) \in \mathbb{Z}_2.$$

The idea is to define $\eta_V(p,q)$ as the number (modulo 2) of trajectories of V going from p to q. We will show later on that this number is indeed finite.

Let us now employ these ideas on the examples previously shown on Section §II:

• *The Height on the Round Sphere:* Here, $C_0 = \mathbb{Z}_2$, with generator a, $C_1 = 0$ and $C_2 = \mathbb{Z}_2$, with generator b (refer to the examples on Section §II to see what these generators are). The homology group is given by

$$H_* = \begin{cases} \mathbb{Z}_2 & \text{if } * = 0,2, \\ 0 & \text{if } * \neq 0,2. \end{cases}$$

The same computation with the height function on the unit n-sphere in $\mathbb{R}^{n+1}$, which also has only a minimum and a maximum, gives

$$H_* = \begin{cases} \mathbb{Z}_2 & \text{if } * = 0,n, \\ 0 & \text{if } * \neq 0,n. \end{cases}$$

• *The "Deformed" Sphere:* Now $C_0 = \mathbb{Z}_2$, with generator a, $C_1 = \mathbb{Z}_2$, with generator b, and $C_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators c and d. By counting the trajectories connecting the critical points, we find

$$\partial c = b, \quad \partial d = b, \quad \text{and} \quad \partial b = 2a = 0.$$

Thus we get

$$\begin{cases} H_0 = \mathbb{Z}_2, \\ H_1 = 0, \\ H_2 = \mathbb{Z}_2. \end{cases}$$

Note that even though the complex is quite different from that of the regular sphere, its homology is the same (as expected).

• *The Torus:* Consider the function $\cos(2\pi x) + \cos(2\pi y)$ on the torus. Here $C_0 = \mathbb{Z}_2$, with generator a, $C_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators b and c, and $C_2 = \mathbb{Z}_2$, with generator d. The differentials are

$$\partial d = 2b + 2c = 0 \qquad \text{and} \qquad \partial b = \partial c = 2a = 0.$$

Thus the homology modulo 2 is

$$\begin{cases} H_0 = \mathbb{Z}_2, \\ H_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\ H_2 = \mathbb{Z}_2, \end{cases}$$

which is indeed the homology of $\mathbb{T}^2$.

If p and q are two critical points of a function $f: M \to \mathbb{R}$, then we let $\mathcal{L}_V(p,q)$ denote the set of trajectories from p to q of the vector field V. Similarly, the set of broken trajectories from p to q is given by

$$\overline{\mathcal{L}}_V(p,q) = \bigcup_{c_i \in \mathrm{Crit}(f)} \mathcal{L}_V(p,c_1) \times \cdots \times \mathcal{L}_V(c_{n-1},q).$$

(As suggested by the notation, this space is meant to be a compactification of $\mathcal{L}_V(p,q)$.)[3]

Now, if p is a critical point of the Morse function f, then its stable manifold $W^s(p)$, which is diffeomorphic to a disk, is an orientable manifold. We then choose, for each critical point $p \in M$, an orientation for $W^s(p)$, for which there is a corresponding co-orientation on $W^u(p)$. Thus, if p and q are any two critical points, we have that $W^u(p) \pitchfork W^s(q)$ is therefore oriented. The same holds for its intersection with a regular level set (a regular level set is co-oriented by the transverse orientation given by the pseudo-gradient field V that is used). Hence the space of trajectories $\mathcal{L}_V(p,q)$ is also an oriented manifold.

Since the homology $H_*(f,X)$ (X being a vector field) of the Morse complex on a manifold V depends only on V, we denote it by $HM_*(V;\mathbb{Z}_2)$ (for "Morse homology modulo 2" of V). Likewise, the homology of the complex taking into account orientations is denoted by $HM_*(V;\mathbb{Z})$ (integral homology of V).

Proposition 2. *If V is a compact connected manifold, then $HM_0(V;\mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Corollary 1. *Let V be a compact connected manifold of dimension n. Then $HM_n(V;\mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Both the proposition and the corollary do in fact hold over $\mathbb{Z}$ if we take into account orientations. The analogue of the corollary states that if V is a compact, connected, oriented n-manifold, then $HM_n(V;\mathbb{Z}) \cong \mathbb{Z}$.

Corollary 2. *Let V be a compact n-manifold. Then $HM_0(V;\mathbb{Z}_2)$ and $HM_n(V;\mathbb{Z}_2)$ are $\mathbb{Z}_2$-vector spaces of dimension the number of connected components of V.*

Proof. Write V as the disjoint union of its connected components $V = \amalg_{j=1}^k V_j$. Then on each V_j, choose a Morse function f_j and a suitable vector field X_j. It then follows that

$$C_*\left(\amalg f_j\right) = \oplus C_*(f_j) \quad \text{and} \quad \partial_{\amalg X_j} = \oplus \partial X_j.$$

□

Proposition 3. *If the manifold V admits a Morse function with no critical points of index 1, then it is simply connected.*

Proof. We may assume that V is path-connected and choose a minimum (say p_0) of f as the base point. Let α be a loop in V. We may also assume that this loop is smooth. If q is a critical point of index k, then we know that $\dim W^s(q) = n - k$. By a general position argument, we may further assume that α meets none of the stable manifolds of the critical points of index greater than or equal to 2. Since we have assumed that there are no critical points of index 1, the loop α is contained in the union of the stable manifolds of the local minima, which are disjoint. Therefore α

[3] See [AD, Section 3.2] for details.

is contained in one of these stable manifolds, namely that of the minimum p_0. But this is a disk, and therefore α is contractible onto the base point p_0. □

We are now going to compare the fundamental group and the first homology group of a connected manifold V. First we state the following fundamental result:

Theorem 1. *Let V be a compact connected manifold of dimension 1. Then V is diffeomorphic to $\mathbb{S}^1$ if $\partial V = \emptyset$ and diffeomorphic to $[0,1]$ if its boundary is nontrivial.*

Proposition 4. *If V is simply connected, then $HM_1(V;\mathbb{Z}_2) = 0$.*

Proof. We choose a Morse-Smale pair (f, X) (meaning a Morse function f and a generic (i.e. Smale) pseudo-gradient field X) on V. Then we begin by describing the 1-cycles in V; that is, the elements α of $C_1(f)$ such that $\partial_X \alpha = 0$. These are linear combinations $\alpha = a_1 + \cdots + a_k$ of critical points of index 1, where $\partial_X a_1 + \cdots + \partial_X a_k = 0$. Now, for a critical point a of index 1, we have

$$\partial_X a = c_1 + c_2,$$

where c_1 and c_2 are two critical points of index 0, which are not necessarily distinct (see Figure 2): the unstable manifold of a is an open disk of dimension 1 and we continue applying Theorem 1.

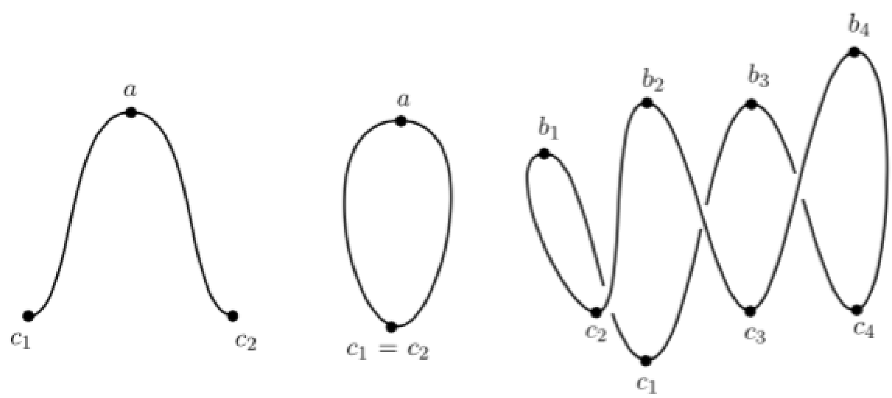


Figure 2

Therefore our cycle α is the sum of cycles

$$\beta = b_1 + \cdots + b_\ell$$

with $\partial b_i = c_i + c_{i+1}$ for local minima $c_1, \ldots, c_\ell$ (with $c_{\ell+1} = c_1$), as in Figure 2. Such a β defines an embedding ι_β of the circle $\mathbb{S}^1$ into V such that the function $g_\beta = f \circ \iota_\beta$ has the c_i as local minima and the b_i as local maxima. Each β in a basis of $\ker \partial \subset C_1(f)$ defines an injective morphism of complexes

$$(\iota_\beta)_* \colon C_*(g_\beta, X_\beta) \to C_*(f, X),$$

where X_β is the vector field on $\mathbb{S}^1$ whose image is the restriction of X, such that the image of

$$(\iota_\beta)_* \colon HM_1(\mathbb{S}^1;\mathbb{Z}_2) \to HM_1(V;\mathbb{Z}_2)$$

is the subspace generated by the class of β.

Now, under the condition of simple connectedness on V, every map $\nu\colon \mathbb{S}^1 \to V$ extends to a map $\widetilde{\nu}\colon \mathbb{D}^2 \to V$. In particular, in homology,

$$\iota_\beta\colon \mathbb{S}^1 \hookrightarrow \mathbb{D}^2 \xrightarrow{\widetilde{\iota}_\beta} V$$

gives a factorization

$$(\iota_\beta)_*\colon HM_1(\mathbb{S}^1;\mathbb{Z}_2) \to HM_1(\mathbb{D}^2;\mathbb{Z}_2) \xrightarrow{(\widetilde{\iota}_\beta)_*} HM_1(V;\mathbb{Z}_2),$$

so that the class of β comes from $HM_1(\mathbb{D}^2;\mathbb{Z}_2) = 0$ and is therefore trivial, as desired. □

We now have gathered (to some extent, at least) enough machinery to delve deep into the construction of the Floer homology, with the aim of proving the Arnold conjecture. This states that the number of periodic trajectories of period 1 of a Hamiltonian vector field on a symplectic manifold W is greater than or equal to $\sum_j HM_j(W;\mathbb{Z}_2)$. A follow-up of this paper will be devoted entirely to this topic.

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