# Differential Forms in Physics II

## Maxwell's Equations

These notes are a follow-up to the previous notes on Stokes's Theorem and differential forms; refer to that paper for definitions and discussions that will be useful here. In this monograph we rewrite Maxwell's Equations in the language of differential forms, showcasing yet again (as in the Stokes's Theorem paper) the elegance and usefulness of the latter. Being verse in both math and physics lingo can be quite advantageous, and this is in fact the point we are trying to drive home through this series of brief monographs.

Our starting point is Maxwell's Equations, as they are written in elementary electrodynamics treatments (in geometrized Gaussian units ($c = \mu_0 = \epsilon_0 = 1$)):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \text{(Gauss)} \tag{1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1b}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{(Faraday)} \tag{1c}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{J} \quad \text{(Ampère)} \tag{1d}$$

where the 3-vectors $\vec{E}$ and $\vec{B}$ are the *electric* and *magnetic field*, respectively, and $\rho$ and $\vec{J}$ are the *charge density* and *current density*, respectively. Now, Maxwell's Equations are the correct theory of electricity and magnetism (and therefore of light), and Special Relativity was invented to explain certain properties of light; whence these equations are relativistically correct. However, one major drawback is the equations' dependence on the choice of a frame (i.e., chart, in math lingo), since neither $\vec{E}$ nor $\vec{B}$ are frame-independent, geometric entities in four-dimensional spacetime. What we need then is a purely geometric object that transcends coordinates and reference frames (i.e., a tensor field) that encapsulates both $\vec{E}$ and $\vec{B}$. Hence we introduce the ***electromagnetic tensor field*** (often referred to as the ***Faraday tensor field***), $\tilde{\boldsymbol{F}}$, as a $\binom{0}{2}$ tensor field

$$\tilde{\boldsymbol{F}} = F_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{2}$$

Following the abstract index notation tradition, it is customary to write $\tilde{\boldsymbol{F}}$ as $F_{ab}$. [1] The components of $F_{ab}$ on a Lorentz frame (i.e., a Riemann normal coordinates chart) $\{x^\mu\}$ are given by [2]

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{3}$$

where $(E^1, E^2, E^3)$ and $(B^1, B^2, B^3)$ are the components of the 3-vectors $\vec{E}$ and $\vec{B}$, respectively. The corresponding contravariant $\binom{2}{0}$ tensor field $F^{ab} = g^{ac}g^{bd}F_{cd}$ has components $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\beta\nu}F_{\alpha\beta}$ in

[1] In this notation, letters $a - h$ and $o - z$ are used for 4-dimensional spacetime indices that run from 0 to 3, whereas the letters $i - n$ are reserved for 3-dimensional spatial indices that run from 1 to 3. Lowercase Greek letters are reserved for components in a chosen basis (see, e.g., Wald's text for reference).

[2] Be aware that, as customary in relativistic physics, our Lorentzian signature of the metric tensor is $(- + ++)$; in electrodynamics and particle physics references the usual signature is $(+ - --)$.

a Lorentz frame given by [3,4]

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}. \tag{4}$$

It is obvious from (3) and (4) that both $F_{ab}$ and $F^{ab}$ are antisymmetric; i.e., $F_{ab} = -F_{ba}$ and $F^{ab} = -F^{ba}$. In particular, $F_{ab}$ is a 2-form, and so we may rewrite (2) as

$$\tilde{\boldsymbol{F}} = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = F_{|\mu\nu|} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{5}$$

(Vertical bars wrapping the indices indicate that we are summing over a strictly-increasing sequence; refer to the first paper for clarification on the notation.)

Note that, in terms of the electromagnetic tensor field, the divergence $\partial_\nu F^{\mu\nu}$ yields two of the original Maxwell's equations:

$$\begin{aligned} \partial_\nu F^{0\nu} = \partial_0 \overbrace{F^{00}}^{=0} + \partial_i F^{0i} &= \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \\ &= \partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 \\ &= \partial_i E^i. \end{aligned}$$

$$\begin{aligned} \partial_\nu F^{i\nu} &= \partial_0 F^{i0} + \partial_j F^{ij} \\ &= \partial_t(-E^i) + \partial_1 F^{i1} + \partial_2 F^{i2} + \partial_3 F^{i3} \\ &= -\partial_t E^i \underbrace{-\partial_1 B^3 + \partial_1 B^2 + \partial_2 B^3 - \partial_2 B^1 - \partial_3 B^2 + \partial_3 B^1}_{=\hat{\epsilon}^{ij}{}_k B^k{}_{,j}} \\ &= \hat{\epsilon}^{ij}{}_k B^k{}_{,j} - \partial_t E^i. \end{aligned}$$

But $\partial_i E^i$ and $\hat{\epsilon}^{ij}{}_k B^k{}_{,j} - \partial_t E^i$ are precisely the index-notation versions of $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{B} - \partial_t \vec{E}$, respectively. [5] Thus, introducing the 4-vector $J^a = (\rho, \vec{J})$, we have rewritten two of Maxwell's Equations ((1a) and (1d)) as

$$\partial_b F^{ab} = 4\pi J^a,$$

and consequently, via minimal coupling,

$$\boxed{\nabla_b F^{ab} = 4\pi J^a} \tag{6}$$

[3] $g_{ab}$ is the Lorentzian metric of curved spacetime (General Relativity); $\eta_{ab}$ is the Minkowskian metric of spacetime (Special Relativity).

[4] Having chosen a Lorentz frame, we can set $g_{\mu\nu}(\mathcal{P}) = \eta_{\mu\nu}(\mathcal{P})$, *at some specific event* (i.e., at some specific point) $\mathcal{P}$. This *Equivalence Principle* is loosely stated in physics lingo as "In any local inertial frame (i.e., a Lorentz frame) the physical laws must reduce to those in Minkowski spacetime" (this principle suggests the simple rule known as *minimal coupling*; $\eta_{ab} \leftrightarrow g_{ab}$, $\partial_a \leftrightarrow \nabla_a$). This is as good as it gets in a general curved spacetime; in a neighborhood of $\mathcal{P}$ it is not true that $g_{\mu\nu} = \eta_{\mu\nu}$, only at $\mathcal{P}$. Loosely speaking, in a general curved spacetime "gravity never truly goes away." More rigorously, what this means is that, even in normal coordinates where at $\mathcal{P}$ the Christoffel symbols vanish ($\Gamma^a_{bc}(\mathcal{P}) = 0$), the gradients of these symbols in a neighborhood of $\mathcal{P}$ do not completely vanish in curved spacetimes ($\partial_d \Gamma^a_{bc} \neq 0$). As a consequence the curvature never truly vanishes in curved spacetimes, even when choosing local Lorentz frames ($R^d{}_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab}$).

[5] $\hat{\epsilon}^{ij}{}_k$ may look weird to the reader in this form; rest assured that we are not "lowering an index" in the traditional sense (i.e., contracting with the metric tensor), since $\hat{\epsilon}$ is not a tensor. This is merely a convenient notation consistent with Einstein summation.

As for the remaining two equations, consider expanding $\partial_{[\mu}F_{\nu\rho]}$,

$$\begin{aligned} 3\partial_{[\mu}F_{\nu\rho]} &= 3\cdot\frac{1}{6}\left(\partial_\mu F_{\nu\rho}+\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu}-\partial_\nu F_{\mu\rho}-\partial_\rho F_{\nu\mu}-\partial_\mu F_{\rho\nu}\right)\\ &= 3\cdot\frac{1}{6}\left(2\partial_\mu F_{\nu\rho}+2\partial_\rho F_{\mu\nu}+2\partial_\nu F_{\rho\mu}\right)\\ &= \partial_\mu F_{\nu\rho}+\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu}. \end{aligned}$$

For clarity, we first consider the spatial part (set, say, $\mu = 1, = 2, \rho = 3$):

$$\partial_1 F_{23}+\partial_3 F_{12}+\partial_2 F_{31} = \partial_1 B^1+\partial_3 B^3+\partial_2 B^2 = \partial_i B^i.$$

Now include one temporal index (say, $\mu = 0$):

$$\begin{aligned} \partial_0 F_{ij}+\partial_j F_{0i}+\partial_i F_{j0} &= \partial_0 F_{12}+\partial_0 F_{23}+\partial_0 F_{31}+\partial_2 F_{01}+\partial_3 F_{02}+\partial_1 F_{03}+\partial_1 F_{20}+\partial_2 F_{30}+\partial_3 F_{10}\\ &= \partial_t B^3+\partial_t B^1+\partial_t B^2-\partial_2 E^1-\partial_3 E^2-\partial_1 E^3+\partial_1 E^2+\partial_2 E^3+\partial_3 E^1\\ &= \partial_t B^i+\hat{\epsilon}^{ij}{}_k E^k{}_{,j}. \end{aligned}$$

But from (1b), $\partial_i B^i = 0$, and from (1c), $\partial_t E^i+\hat{\epsilon}^{ij}{}_k E^k{}_{,j} = 0$; hence

$$3\partial_{[\mu}F_{\nu\rho]} = \partial_\mu F_{\nu\rho}+\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu} = \underbrace{\partial_i B^i}_{=0}+\underbrace{\partial_t B^i+\hat{\epsilon}^{ij}{}_k E^k{}_{,j}}_{=0} = 0.$$

Thus, we have

$$\partial_{[a}F_{bc]} = 0$$

and consequently, via minimal coupling,

$$\boxed{\nabla_{[a}F_{bc]} = 0} \tag{7}$$

So we have rewritten the original Maxwell's Equations (1) as two manifestly covariant, nice-looking equations ((6) & (7)). The latter are the form most widely used in relativistic physics treatments; however, our goal here is to link mathematics and physics by rewriting these equations in the language of differential forms. The reader may wonder why the need to go further; after all, it is undeniable that equations (6) & (7) are already in the most convenient possible form for calculations in a chosen set of coordinates. However, the greatest physical insights do not always (in fact, rarely) come from plugging and chugging components and analyzing the results; a broader, more abstract view may very well help to spot potentially hidden details. For instance, electromagnetism itself is a *gauge theory*, and the geometrical interpretation of such theories is very much a point of emphasis in contemporary, advanced treatments of theoretical physics. Any physicist working on gauge theories knows how crucial it is to be verse on the mathematical language of differential forms, fiber bundles, Lie algebras and Lie groups, etc ...

Without further ado, let us write the equations' final mold in terms of differential forms. Upon close inspection of (7) and its derivation, we notice that this is in fact the index-notation version of the differential $\mathrm{d}\tilde{\boldsymbol{F}}$: For any $k$-form $\omega$, its differential $\mathrm{d}\omega$ in index-notation is given by

$$(\mathrm{d}\omega)_{a_1\ldots a_{k+1}} = (k+1)\partial_{[a_1}\omega_{a_2\ldots a_{k+1}]}.$$

In our case,

$$(\mathrm{d}F)_{abc} = 3\partial_{[a}F_{bc]} = 0 \tag{8}$$

which, in index-free-notation is precisely

$$\boxed{\mathrm{d}\tilde{\boldsymbol{F}} = 0} \tag{7 ⊛}$$

However, let us not be lazy and derive (7 ⊛) using mathematical (index-free) notation, for the mathematically-inclined reader who is still not very comfortable with index gymnastics (be sure to refer back to Part I (Stokes's Theorem) if something in the notation is unclear). First a warning regarding the orientation of differential forms in Lorentzian signature: In spacetime, the temporal index always goes first given a choice of (right-handed or left-handed) spatial orientation; time-orientation is a whole different ball of wax. For example, whilst the reader might be tempted to claim that $\mathrm{d}x^3 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1$ is a correct right-handed orientation, (s)he must keep in mind that $x^0$ is a time coordinate, so the only indices that must be considered for the orientation are the spatial ones (in this example the correct orientation is therefore $\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1$).

Let us get right to it; from (5),

$$\begin{aligned}
\mathrm{d}\tilde{\boldsymbol{F}} &= \mathrm{d}\left(F_{|\mu\nu|}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right)\\
&= \partial_\rho F_{|\mu\nu|}\mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\\
&= \partial_0 F_{12}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \partial_0 F_{23}\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_0 F_{13}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3\\
&\quad + \partial_1 F_{23}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_1 F_{02}\,\mathrm{d}x^1 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^2 + \partial_1 F_{03}\,\mathrm{d}x^1 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^3\\
&\quad + \partial_2 F_{01}\,\mathrm{d}x^2 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 + \partial_2 F_{03}\,\mathrm{d}x^2 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^3 + \partial_2 F_{13}\,\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3\\
&\quad + \partial_3 F_{01}\,\mathrm{d}x^3 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 + \partial_3 F_{02}\,\mathrm{d}x^3 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^2 + \partial_3 F_{12}\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\\
&= (\partial_0 F_{12} - \partial_1 F_{02} + \partial_2 F_{01})\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + (\partial_0 F_{23} - \partial_2 F_{03} + \partial_3 F_{02})\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\\
&\quad + (-\partial_0 F_{13} + \partial_1 F_{03} - \partial_3 F_{01})\,\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + (\partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\\
&= \left(\partial_t B^3 + \partial_1 E^2 - \partial_2 E^1\right)\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_t B^1 + \partial_2 E^3 - \partial_3 E^2\right)\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\\
&\quad + \left(\partial_t B^2 - \partial_1 E^3 + \partial_3 E^1\right)\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \left(\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.
\end{aligned}$$

The green part is the divergence $\partial_i B^i$, which vanishes from (1b), whilst the orange part is precisely the components of $\partial_t B^i + \hat{\epsilon}^{ij}{}_k E^k{}_{,j}$, which also vanish ((1c)). Hence, the lengthy algebra above proves (7 ⊛), using the differential forms treatment. □

⚠ **Word of Warning** ⚠ Something quite irritating that (unfortunately) is quite ubiquitous in the physics literature, is the claim of the "existence of some *electric potential*" $\tilde{\boldsymbol{A}}$ whose differential is $\tilde{\boldsymbol{F}}$; i.e., $\tilde{\boldsymbol{F}} = \mathrm{d}\tilde{\boldsymbol{A}}$ ($F_{ab} = 2\partial_{[a}A_{b]}$, in coordinates). Usually the potential $\tilde{\boldsymbol{A}}$ is introduced and then equations such as $\mathrm{d}\tilde{\boldsymbol{F}} = 0$ follow trivially from the property that $\mathrm{d} \circ \mathrm{d} = 0$. Howevever, there is one glaring problem with this argument, and if you followed closely our discussion of *exact* and *closed* forms on Part I you will immediately see the issue: whilst $\tilde{\boldsymbol{F}}$ is indeed closed, as we just proved with the messy algebra above, the Poincaré Lemma states that every closed form is exact *only* in contractible spaces. If we are dealing with a simply-connected manifold (as per usual in electrodynamics; Minkowski spacetime), then the aforementioned assumption of an electric potential is not ill-founded. Moreover, if we are considering a local problem, then global topological considerations can also be disregarded (i.e., the existence of such potential is always guaranteed *locally*, regardless of the manifold's topology). However, since the existence of such potential does not hold water in a general manifold (say, a manifold "with holes"; e.g., a circle or a torus) we do not find this claim to be good taste and abstain from it.

All that remains now is to show the validity of the inhomogeneous equations (6) in terms of differential forms. In what follows we shall consider the Levi-Civita tensor $\boldsymbol{\epsilon}$ (not the symbol $\hat{\boldsymbol{\epsilon}}$ we used in Part I; this is the actual tensor $\boldsymbol{\epsilon} \equiv \sqrt{|g|}\hat{\boldsymbol{\epsilon}}$), given in coordinates by [6]

$$\epsilon_{\mu_1\ldots\mu_k} = \begin{cases} +1 & \text{if } (\mu_1 \ldots \mu_k) \text{ is an even permutation of } 1, \ldots, k, \\ -1 & \text{if } (\mu_1 \ldots \mu_k) \text{ is an odd permutation of } 1, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

[6] $g \equiv \det \boldsymbol{g}$ (= $\det g_{ab}$, in abstract index lingo); this is fairly standard notation.

This being a tensor, its indeces can be raised/lowered with the metric; thus,

$$\boldsymbol{\epsilon}^{-1} = \epsilon^{\mu_1\dots\mu_k} = g^{\mu_1\nu_1}\dots g^{\mu_k\nu_k}\,\epsilon_{\nu_1\dots\nu_k} = \boldsymbol{g}^{-1}\boldsymbol{\epsilon}.$$

**Definition 1.** *Let $(\mathcal{M}, \boldsymbol{g})$ be a (pseudo)Riemannian n-dimensional manifold $\mathcal{M}$ with metric tensor $\boldsymbol{g}$, and let $\omega \in \Omega^k(\mathcal{M})$. Then* ***Hodge star operator****, $\star$, is a map $\star\colon \Omega^k(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$, given by*[7]

$$\begin{aligned}\star\omega &= \star\left(\frac{1}{k!}\omega_{\mu_1\dots\mu_k}\,\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_k}\right)\\ &= \frac{1}{k!\,(n-k)!}\omega_{\mu_1\dots\mu_k}\,g^{\mu_1\nu_1}\dots g^{\mu_k\nu_k}\,\epsilon_{\nu_1\dots\nu_n}\,\mathrm{d}x^{\nu_{k+1}}\wedge\dots\wedge\mathrm{d}x^{\nu_n}\\ &= \frac{1}{k!\,(n-k)!}\omega_{\mu_1\dots\mu_k}\,\epsilon^{\mu_1\dots\mu_k}{}_{\nu_{k+1}\dots\nu_n}\,\mathrm{d}x^{\nu_{k+1}}\wedge\dots\wedge\mathrm{d}x^{\nu_n}.\end{aligned} \tag{9}$$

*Thus, in a chart $\{x^\mu\}$,*

$$(\star\omega)_{\mu_1\dots\mu_{n-k}} = \frac{1}{k!}\epsilon^{\nu_1\dots\nu_k}{}_{\mu_1\dots\mu_{n-k}}\omega_{\nu_1\dots\nu_k}. \tag{10}$$

Thus, the Hodge operator takes $k$-forms to their dual $(n-k)$-forms in an $n$-dimensional (pseudo) Riemannian manifold. In our case, $n = 4$ and $\tilde{\boldsymbol{F}}$ is a 2-form, so its dual $\star\tilde{\boldsymbol{F}}$ is also a 2-form. Let us now expand $\star\tilde{\boldsymbol{F}}$ and see what it looks like in coordinates:

$$(\star F)_{\mu\nu} = \frac{1}{2}\epsilon^{\rho\sigma}{}_{\mu\nu}\,F_{\rho\sigma} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\,g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\,F^{\alpha\beta}.$$

From the antisymmetry of both $\boldsymbol{\epsilon}$ and $\boldsymbol{F}$, we gather that $\epsilon_{\alpha\beta\mu\nu}\,F^{\alpha\beta} = \epsilon_{\beta\alpha\mu\nu}\,F^{\beta\alpha}$; we will use this below. Now consider first the temporal components,

$$\begin{aligned}(\star F)_{01} &= \frac{1}{2}\epsilon_{\alpha\beta01}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{2301}\,F^{23}\right) = \epsilon_{0123}\,F^{23} = F^{23}\\ (\star F)_{02} &= \frac{1}{2}\epsilon_{\alpha\beta02}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{1302}\,F^{13}\right) = -\epsilon_{0123}\,F^{13} = -F^{13}\\ (\star F)_{03} &= \frac{1}{2}\epsilon_{\alpha\beta03}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{1203}\,F^{12}\right) = \epsilon_{0123}\,F^{12} = F^{12},\end{aligned}$$

and now the spatial components,

$$\begin{aligned}(\star F)_{12} &= \frac{1}{2}\epsilon_{\alpha\beta12}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{0312}\,F^{03}\right) = \epsilon_{0123}\,F^{03} = F^{03}\\ (\star F)_{13} &= \frac{1}{2}\epsilon_{\alpha\beta13}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{0213}\,F^{02}\right) = -\epsilon_{0123}\,F^{02} = -F^{02}\\ (\star F)_{23} &= \frac{1}{2}\epsilon_{\alpha\beta23}\,F^{\alpha\beta} = \frac{1}{2}\left(2\,\epsilon_{0123}\,F^{01}\right) = F^{01},\end{aligned}$$

and the remaining components follow from the antisymmetry of $\boldsymbol{F}$. Putting all this together, and referring to (4), we have gathered all the components of $\star\tilde{\boldsymbol{F}}$:

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B^1 & B^2 & B^3\\ -B^1 & 0 & E^3 & -E^2\\ -B^2 & -E^3 & 0 & E^1\\ -B^3 & E^2 & -E^1 & 0\end{pmatrix}. \tag{11}$$

Our last order of business is to compute the differential $\mathrm{d}\star\tilde{\boldsymbol{F}}$. As we saw with the calculation of $\mathrm{d}\tilde{\boldsymbol{F}}$ above, the algebra of can get pretty messy ... Fortunately, we have already done all the heavy lifting! As we derived in the proof of (7 ⊛), the differential of a 2-form $\omega$ is given by

$$\begin{aligned}\mathrm{d}\omega = {}&(\partial_0\omega_{12} - \partial_1\omega_{02} + \partial_2\omega_{01})\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + (\partial_0\omega_{23} - \partial_2\omega_{03} + \partial_3\omega_{02})\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\ &+ (-\partial_0\omega_{13} + \partial_1\omega_{03} - \partial_3\omega_{01})\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 + (\partial_1\omega_{23} - \partial_2\omega_{13} + \partial_3\omega_{12})\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3.\end{aligned} \tag{12}$$

[7] The map is also often called the ***Hodge star dual***, or just the ***Hodge dual***.

Plugging $\star\tilde{\boldsymbol{F}}$ in place of $\omega$ yields

$$\begin{aligned}
\mathrm{d}\star\tilde{\boldsymbol{F}} &= (\partial_0(\star F)_{12} - \partial_1(\star F)_{02} + \partial_2(\star F)_{01})\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 \\
&\quad + (\partial_0(\star F)_{23} - \partial_2(\star F)_{03} + \partial_3(\star F)_{02})\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad + (-\partial_0(\star F)_{13} + \partial_1(\star F)_{03} - \partial_3(\star F)_{01})\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 \\
&\quad + (\partial_1(\star F)_{23} - \partial_2(\star F)_{13} + \partial_3(\star F)_{12})\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&= \left(\partial_t E^3 - \partial_1 B^2 + \partial_2 B^1\right)\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \left(\partial_t E^1 - \partial_2 B^3 + \partial_3 B^2\right)\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad + \left(\partial_t E^2 + \partial_1 B^3 - \partial_3 B^1\right)\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 + \left(\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3. \qquad (13)
\end{aligned}$$

The green part is the divergence $\partial_i E^i$, which equals $4\pi\rho$ from (1a), whilst the orange part is precisely the components of $\partial_t E^i - \hat{\epsilon}^{ij}{}_k B^k{}_{,j}$, which equal $-4\pi\vec{J}$ ((1d)). Do not fret about the negative sign; the components on the last equality of (13) are in fact the components of a 3-form, $\star\tilde{\boldsymbol{J}}$, (multiplied by $4\pi$) that is dual (via the Hodge operator) to the 1-form obtained from lowering an index of the 4-current $\boldsymbol{J} = (\rho, \vec{J})$: Using (9),

$$\begin{aligned}
\star\tilde{\boldsymbol{J}} &= \frac{1}{1!\,(4-1)!}\epsilon^{\nu}{}_{\alpha\beta\gamma}\,J_\nu\,\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma \\
&= \frac{1}{6}\epsilon_{\mu\alpha\beta\gamma}\,g^{\mu\nu}J_\nu\,\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma \\
&= \epsilon_{\mu|\alpha\beta\gamma|}\,J^\mu\,\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma,
\end{aligned}$$

where, as we have done before, we use the antisymmetry of a differential form to sum only over a strictly-increasing sequence the indices inside the vertical bars (thus avoiding repeating terms and getting rid of the $1/3!$ coefficient). Hence we have

$$\begin{aligned}
\star\tilde{\boldsymbol{J}} &= \epsilon_{3012}J^3\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \epsilon_{0123}J^0\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad + \epsilon_{1023}J^1\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + \epsilon_{2103}J^2\,\mathrm{d}x^1\wedge\mathrm{d}x^0\wedge\mathrm{d}x^3 \\
&= -\epsilon_{0123}J^3\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \epsilon_{0123}\,\rho\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad - \epsilon_{0123}J^1\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 - \epsilon_{0123}J^2\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 \\
&= -J^3\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \rho\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad - J^1\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 - J^2\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 \\
&= \left(\frac{\partial_t E^3 - \partial_1 B^2 + \partial_2 B^1}{4\pi}\right)\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \left(\frac{\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3}{4\pi}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad + \left(\frac{\partial_t E^1 - \partial_2 B^3 + \partial_3 B^2}{4\pi}\right)\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + \left(\frac{\partial_t E^2 + \partial_1 B^3 - \partial_3 B^1}{4\pi}\right)\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1. \qquad (14)
\end{aligned}$$

Thus, combining (13) and (14), we get the differential-forms-version of the inhomogeneous Maxwell's Equations (6),

$$\boxed{\mathrm{d}\star\tilde{\boldsymbol{F}} = 4\pi\star\tilde{\boldsymbol{J}}} \qquad (6\,\circledast)$$

Hence we have concluded our task. All of our hard work summed up in this tiny box:

$$\boxed{\begin{aligned} \mathrm{d}\tilde{\boldsymbol{F}} &= 0 \\ \mathrm{d}\star\tilde{\boldsymbol{F}} &= 4\pi\star\tilde{\boldsymbol{J}} \end{aligned}}$$

Or, in the vacuum, with no electric source to consider (as per usual in GR),

$$\boxed{\begin{aligned} \mathrm{d}\tilde{\boldsymbol{F}} &= 0 \\ \mathrm{d}\star\tilde{\boldsymbol{F}} &= 0 \end{aligned}}$$