Differential Forms in Physics I

Stokes's Theorem

In what follows we attempt to showcase the elegance and usefulness of the language of differential forms and the exterior derivative as it applies to certain areas of physics. In this first of two papers we present Stokes's Theorem in the language of differential forms, and show how it generalizes the more basic treatments found in elementary vector calculus and undergraduate electrodynamics; the follow-up paper will focus on Maxwell's Equations. This is not an introduction to differential forms; the reader is assumed familiarity with the subject and a certain level of "mathematical maturity" (mathematicians certainly love to throw around this phrase, and I make no exception ☺).

Our starting point is the definition of the wedge product $\wedge$ and of the exterior derivative d. The former is needed because, for a given k-form $\omega \in \Omega^k(\mathcal{M})$ and ℓ-form $\eta \in \Omega^\ell(\mathcal{M})$ in some smooth manifold $\mathcal{M}$, it is not always the case that their tensor product $\omega \otimes \eta$ is in $\Omega^{k+\ell}(\mathcal{M})$, whereas $\omega \wedge \eta \in \Omega^{k+\ell}(\mathcal{M})$.

Definition 1. *Let $\mathcal{M}$ be a smooth manifold, denote by $T^k(\mathcal{M})$ the space of all smooth covariant k-tensor fields on $\mathcal{M}$, and let $\Omega^k(\mathcal{M})$ be the space of all smooth k-forms (smooth, alternating covariant k-tensor fields) on $\mathcal{M}$. Then we can define a projection, named the **alternating map**,* Alt: $T^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ *given by*

$$\mathrm{Alt}(\boldsymbol{T}) = \frac{1}{k!} \sum_{\pi \in S_k} (\operatorname{sgn} \pi)\, {}^{\pi}\boldsymbol{T}, \tag{1}$$

where S_k is the symmetric group on k elements and ${}^{\pi}\boldsymbol{T}$ is the tensor $\boldsymbol{T}$ with a permutation π applied on its indices.

More explicitly, given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathfrak{X}(\mathcal{M})$, equation (1) takes the form

$$(\mathrm{Alt}(\boldsymbol{T}))(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \frac{1}{k!} \sum_{\pi \in S_k} (\operatorname{sgn} \pi) \boldsymbol{T}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}).$$

The notation used thus far is the one commonly used in math. One of our key objectives in these notes is to bridge the gap between math and physics so, to that end, let us switch over to index notation –the one most commonly used in relativistic physics. In index notation equation (1) is written more compactly,

$$(\mathrm{Alt}(\boldsymbol{T}))_{\mu_1 \ldots \mu_k} = T_{[\mu_1 \ldots \mu_k]}, \tag{2}$$

where the brackets denote the antisymmetric parts of $\boldsymbol{T}$; i.e.,

$$T_{[\mu_1 \ldots \mu_k]} = \frac{1}{k!} \hat{\epsilon}^{\mu_1 \cdots \mu_k} T_{\mu_1 \ldots \mu_k}, \tag{3}$$

where we used Einstein summation and the Levi-Civita symbol $\hat{\boldsymbol{\epsilon}}$, which is given by [1]

$$\hat{\epsilon}^{\mu_1 \cdots \mu_k} = \begin{cases} +1 & \text{if } (\mu_1 \ldots \mu_k) \text{ is an even permutation of } 1, \ldots, k, \\ -1 & \text{if } (\mu_1 \ldots \mu_k) \text{ is an odd permutation of } 1, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$$
$$T_{[abc]} = \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{cba} - T_{bac}).$$

[1] Note that this is just a permutation symbol, not the Levi-Civita tensor $\boldsymbol{\epsilon} = \sqrt{g}\hat{\boldsymbol{\epsilon}}$ (in fact, we haven't even introduced a metric yet in our discussion). When using this symbol, since the metric is not involved, there is no change of sign between the covariant and contravariant versions, i.e., $\hat{\epsilon}^{\mu_1 \cdots \mu_k} = \hat{\epsilon}_{\mu_1 \ldots \mu_k}$.

Given the above definition, it is clear that $\mathrm{Alt}(\boldsymbol{T}) = \boldsymbol{T} \iff \boldsymbol{T}$ is an alternating/antisymmetric k-tensor field (i.e., a k-form). When dealing with tensor products of these differential forms, it turns out convenient to work with a slight modification of Alt. To motivate this, let us apply Alt to a tensor product of a k-tensor $\boldsymbol{T}$ and an ℓ-tensor $\boldsymbol{S}$:

$$\mathrm{Alt}(\boldsymbol{T} \otimes \boldsymbol{S})_{\mu_1 \dots \mu_{k+\ell}} = T_{[\mu_1 \dots \mu_k} S_{\mu_{k+1} \dots \mu_{k+\ell}]} = \frac{1}{(k+\ell)!} \hat{\epsilon}^{\mu_1 \dots \mu_{k+\ell}} T_{\mu_1 \dots \mu_k} S_{\mu_{k+1} \dots \mu_{k+\ell}}.$$

The factor $(k+\ell)!$ in the denominator is a bit inconvenient for calculations (the adopted convention is, of course, a matter of personal preference); the modification to Alt, called the *wedge product*, gets rid of this factor. It is introduced only for smooth forms.

Definition 2. *The **wedge product** of a k-form $\omega \in \Omega^k(\mathcal{M})$ and an ℓ-form $\eta \in \Omega^\ell(\mathcal{M}$ is a map $\wedge\colon \Omega^k(\mathcal{M}) \times \Omega^\ell(\mathcal{M}) \to \Omega^{k+\ell}(\mathcal{M})$, given by*

$$\wedge(\omega, \eta) = \omega \wedge \eta \equiv \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta). \tag{4}$$

Thus, in a chart $\{x^i\}$, a general k-form ω is expressed as

$$\omega = \frac{1}{k!} \omega_{j_1 \dots j_k} \, \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k} \equiv \omega_{|j_1 \dots j_k|} \, \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k},$$

where (as we shall always do) we used Einstein summation, and the vertical bars placed on the indices in the second equality indicates that the sum is to be done over a strictly-increasing sequence $J = j_1, \dots, j_k$ with $j_i > j_{i-1}$. This is justified by the fact that $\mathrm{d}x^{j_i} \wedge \mathrm{d}x^{j_k} = -\mathrm{d}x^{j_k} \wedge \mathrm{d}x^{j_i}$ and $\mathrm{d}x^{j_i} \wedge \mathrm{d}x^{j_i} = 0$.

Definition 3. *Let $\mathcal{M}$ be a smooth manifold, denote by $\Omega^k(\mathcal{M})$ the space of all smooth k-forms on $\mathcal{M}$, and consider a strictly-increasing sequence $J = j_1, \dots, j_k$ with $j_i > j_{i-1}$. The **exterior derivative** is a map $\mathrm{d}\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ that satisfies, for any k-form ω,*

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}(\omega_J \mathrm{d}x^J) &= \mathrm{d}\left(\omega_{|j_1 \dots j_k|} \, \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k}\right) \\ &= \partial_i \omega_{|j_1 \dots j_k|} \mathrm{d}x^i \wedge \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k}. \end{aligned} \tag{5}$$

Example 1. *If ω is a 1-form, (5) yields* [2]

$$\begin{aligned} \mathrm{d}\omega = \mathrm{d}(\omega_j \mathrm{d}x^j) &= \sum_{i,j} \partial_i \omega_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i<j} \partial_i \omega_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j + \sum_{i>j} \partial_i \omega_j \, \underbrace{\mathrm{d}x^i \wedge \mathrm{d}x^j}_{=-\mathrm{d}x^j \wedge \mathrm{d}x^i} \\ &= \sum_{i<j} \left(\partial_i \omega_j - \partial_j \omega_i\right) \, \mathrm{d}x^i \wedge \mathrm{d}x^j. \end{aligned}$$

Theorem 1 (Existence and Uniqueness of Exterior Differentiation). *Suppose $\mathcal{M}$ is a smooth manifold with or without boundary. There are unique operators $\mathrm{d}\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ for all k, called **exterior differentiation**, satisfying the following four properties:*

[2] For clarity, on this particular example we use explicit summation.

- d *is linear over* $\mathbb{R}$.
- *If* $\omega \in \Omega^k(\mathcal{M})$ *and* $\eta \in \Omega^\ell(\mathcal{M})$, *then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \tag{6}$$

- *For* $f \in \Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$, df *is the differential of* f, *given by* $df(X) = Xf$.
- $d \circ d = 0$.

In any smooth coordinate chart, d is given by (5).

Definition 4. *A k-form* ω *is said to be*

- ***exact*** *if* ω *is the exterior derivative of some* $(k-1)$*-form* η, *i.e.,* $\omega = d\eta$;
- ***closed*** *if the exterior derivative of* ω *vanishes, i.e.,* $d\omega = 0$. [3]

Note that, by the last property of Theorem 1, for any k-form θ we have $d^2\theta = d(d\theta) = 0$; thus every exact form is closed. The question of whether every closed form is exact is answered by the *Poincaré Lemma*, which states that in a star-shaped (i.e., a contractible) domain every closed form is indeed exact (for general domains this result fails).

Theorem 2 (Stokes's Theorem). *Let* $\mathcal{M}$ *be an oriented smooth n-manifold with boundary, and let* ω *be a compactly supported smooth* $(n-1)$*-form on* $\mathcal{M}$. *Then*

$$\boxed{\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega} \tag{7}$$

Example 2. *Let* $\mathcal{M}$ *be a smooth manifold and suppose* $\gamma\colon [a,b] \to \mathcal{M}$ *is a smooth embedding, so that the image* $S = \gamma([a,b])$ *is an embedded* 1*-submanifold with boundary in* $\mathcal{M}$. *If we give* S *the orientation such that* γ *is orientation-preserving, then for any smooth function* $f \in C^\infty(\mathcal{M})$, *Stokes's theorem says that*

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)). \tag{8}$$

The following corollaries are straightforward consequences of Stokes's Theorem:

Corollary 1 (Integrals of Exact Forms). *If* $\mathcal{M}$ *is a compact oriented smooth manifold without boundary, then the integral of every exact form over* $\mathcal{M}$ *vanishes:*

$$\int_{\mathcal{M}} d\omega = 0 \quad \textit{if} \quad \partial\mathcal{M} = \emptyset. \tag{9}$$

Corollary 2 (Integrals of Closed Forms over Boundaries). *Suppose* $\mathcal{M}$ *is a compact oriented smooth manifold with boundary. If* ω *is a closed form on* $\mathcal{M}$, *then the integral of* ω *over* $\partial\mathcal{M}$ *vanishes:*

$$\int_{\partial\mathcal{M}} \omega = 0 \quad \textit{if} \quad d\omega = 0 \quad \textit{on } \mathcal{M}. \tag{10}$$

[3] Note that due to the result from Example 1, one often finds in the literature that a 1-form is closed if it satisfies $\partial_i \omega_j = \partial_j \omega_i$.

Equation (7) showcases the elegance of differential forms; we illustrate this elegance/usefulness further in the following discussion:

The Euclidean metric ${}^E g_{ij}$ on $\mathbb{R}^3$ yields an index-lowering isomorphism $\flat\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ (usually called the *flat isomorphism* in the math literature) given by $\flat(X^j) = {}^E g_{ij} X^j = X_i$ for any vector field $X^i \in \mathfrak{X}(\mathbb{R}^3)$. [4] Just as exterior differentiation *increases* the rank of the differential form by one, there is another important operation on differential forms that *decreases* the rank by one, namely the ***interior multiplication*** $i_v\colon \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ by a vector field $\boldsymbol{v} \equiv v^i$; this operation is given by

$$i_v\omega(\boldsymbol{w}_1, \dots, \boldsymbol{w}_{k-1}) = \omega(\boldsymbol{v}, \boldsymbol{w}_1, \dots, \boldsymbol{w}_{k-1}), \tag{11}$$

where $\omega \in \Omega^k(\mathcal{M})$ and $\boldsymbol{v}, \boldsymbol{w}_1, \dots, \boldsymbol{w}_{k-1} \in \mathfrak{X}(\mathcal{M})$. In other words, $i_v\omega$ is obtained from ω by inserting $\boldsymbol{v}$ into the first slot. [5] We use this interior multiplication to construct another map $\beta\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ given by

$$\beta(\boldsymbol{X}) = i_{\boldsymbol{X}}(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3). \tag{12}$$

Lastly, we define another smooth bundle isomorphism $*\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by

$$*(f) = f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{13}$$

The relationships amongst all of these operators and how they relate to d are summarized in the following commutative diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3).
\end{tikzcd}$$

The language of elementary vector calculus requires three different operators (grad, curl, and div) to represent operations that merely require one operator (d) in the language of differential forms. [6] [7] This is illustrated in the vector calculus versions of Stokes's Theorem: For some smooth vector field $\boldsymbol{A} = A^i \in \mathfrak{X}(\mathbb{R}^3)$,

- for $n = 2$ we have

$$\int_{\mathcal{V}_2} \mathrm{curl}(\boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{\Sigma} = \int_{\partial\mathcal{V}_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}, \tag{14}$$

  where $\mathcal{V}_2 \subseteq \mathbb{R}^2$ is a compact bounded region, $\partial\mathcal{V}_2$ is the 1-dimensional closed curve that bounds it, and the last integral is a line integral around that curve. Also, $\mathrm{d}\boldsymbol{\Sigma}$ is the infinitesimal vectorial surface area on $\mathcal{V}_2$.

- for $n = 3$ (this case is usually referred to in the physics literature as *Gauss's Theorem*) we have

$$\int_{\mathcal{V}_3} \mathrm{div}(\boldsymbol{A})\,\mathrm{d}V = \int_{\partial\mathcal{V}_3} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\Sigma}, \tag{15}$$

  where $\mathrm{d}V = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ (usually just written $\mathrm{d}V = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$) is the volume element, $\mathcal{V}_3 \subseteq \mathbb{R}^3$ is a compact bounded region, $\partial\mathcal{V}_3$ is its closed 2-dimensional boundary surface, and $\mathrm{d}\boldsymbol{\Sigma}$ is the infinitesimal vectorial surface area on $\partial\mathcal{V}_3$.

[4] Of course there is also an an index-raising isomorphism $\sharp\colon \Omega^1(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$ (usually called the *sharp isomorphism*) given by $\sharp(X_j) = {}^E g^{ij} X_j = X^i$. These two isomorphisms are called the *musical isomorphisms* in the math literature.

[5] By convention, we interpret $i_v\omega$ to be zero when ω is a 0-covector (i.e., a number).

[6] Also, curl only makes sense in three dimensions, whereas the generalization (d) applies to any arbitrary dimension.

[7] Also, note how the rules $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$ come from the fundamental property $\mathrm{d} \circ \mathrm{d} = 0$.

We now derive (14) (and leave (15) as a trivial, muscle-flexing exercise to the reader) from the more succinct and elegant (7). Before doing so, we need to point out a nuisance that is usually encountered in both physics and elementary vector calculus texts. In these references it is quite common to ignore the difference between vectors and covectors (1-forms) thereby paying no heed to the placement of indices (up or down). This is partly justified by the fact that in Cartesian coordinates the Euclidean metric leaves intact the components of vectors and covectors, so the musical isomorphisms presented earlier do not have any effect whatsoever on these components. Despite this equality of components in the Euclidean case, such index-placement-agnostic behavior can be a slippery slope, and we do not encourage it. For instance, the infinitesimal vectorial surface area $\mathrm{d}\boldsymbol{\Sigma}$ can be written $\mathrm{d}\boldsymbol{\Sigma} = \boldsymbol{n}\mathrm{d}\Sigma = \boldsymbol{n}\mathrm{d}x^1\mathrm{d}x^2$, where "$\boldsymbol{n}$ is the unit normal to the infinitesimal surface area $\mathrm{d}\Sigma = \mathrm{d}x^1\mathrm{d}x^2$ of the parallelogram spanned by the legs $\mathrm{d}x^1$ and $\mathrm{d}x^2$" ... this is what you would find in an elementary physics text; in reality the parallelogram is actually spanned by the vectors *dual* to $\mathrm{d}x^1$ and $\mathrm{d}x^2$, namely $\boldsymbol{\partial}_1$ and $\boldsymbol{\partial}_2$.

The starting point is to lower the index of $\boldsymbol{A} = A^i$ via the flat isomorphism $\flat$ to work exclusively with differential forms, thus obtaining the 1-form $\tilde{\boldsymbol{A}} = A_i$. From Example 1, the exterior derivative of $\tilde{\boldsymbol{A}}$ is

$$\begin{aligned}
\mathrm{d}\tilde{\boldsymbol{A}} &= \mathrm{d}(A_j \mathrm{d}x^j) \\
&= \sum_{i<j} \left(\partial_i A_j - \partial_j A_i\right) \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \left(\partial_1 A_2 - \partial_2 A_1\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_1 A_3 - \partial_3 A_1\right) \underbrace{\mathrm{d}x^1 \wedge \mathrm{d}x^3}_{=-\mathrm{d}x^3 \wedge \mathrm{d}x^1} + \left(\partial_2 A_3 - \partial_3 A_2\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \left(\partial_1 A_2 - \partial_2 A_1\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_2 A_3 - \partial_3 A_2\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\partial_3 A_1 - \partial_1 A_3\right) \mathrm{d}x^3 \wedge \mathrm{d}x^1 .
\end{aligned}$$

(We are color-coding for a reason; it will be evident soon) Thus, with $\mathcal{M} = \mathcal{V}_2$ and $\omega = \tilde{\boldsymbol{A}}$, the LHS of (7) is

$$\begin{aligned}
\int_{\mathcal{V}_2} \mathrm{d}\tilde{\boldsymbol{A}} &= \int_{\mathcal{V}_2} \left(\partial_1 A_2 - \partial_2 A_1\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad + \cancelto{0}{\int_{\mathcal{V}_2} \left(\partial_2 A_3 - \partial_3 A_2\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3} + \cancelto{0}{\int_{\mathcal{V}_2} \left(\partial_3 A_1 - \partial_1 A_3\right) \mathrm{d}x^3 \wedge \mathrm{d}x^1} \\
&= \int_{\mathcal{V}_2} \left(\partial_1 A_2 - \partial_2 A_1\right) \mathrm{d}x^1 \mathrm{d}x^2,
\end{aligned} \tag{16}$$

where the last two integrals vanish because $\mathrm{d}x^3$ plays no part in the volume form of the 2-surface $\mathcal{V}_2$. (We also dropped the wedge $\wedge$ at the end, as it is customary when writing volume forms.)

On the other hand, from vector calculus and elementary physics we know that $\mathrm{curl}(\boldsymbol{A})$ expands as

$$\underbrace{\mathrm{curl}(\boldsymbol{A})}_{\text{vector calculus lingo}} = \underbrace{\hat{\epsilon}^{ij}{}_k A^k{}_{,j}}_{\text{physics abstract index lingo}} = (\partial_2 A^3 - \partial_3 A^2)\boldsymbol{\partial}_1 + (\partial_3 A^1 - \partial_1 A^3)\boldsymbol{\partial}_2 + (\partial_1 A^2 - \partial_2 A^1)\boldsymbol{\partial}_3 .$$

Note the striking similarity between $\mathrm{d}\tilde{\boldsymbol{A}}$ and $\mathrm{curl}(\boldsymbol{A})$ by looking at their matched colors; they are essentially the same operation, although the former is purely in terms of smooth forms and the latter in terms of their dual vectors. In Cartesian coordinates on Euclidean space $A^i = A_i$ and, moreover, $\boldsymbol{\partial}_k$ is precisely the unit normal $\boldsymbol{n}$ to the infinitesimal surface $\mathrm{d}x^i\mathrm{d}x^j$ $(i \neq j \neq k)$; i.e., for $i \neq j \neq k$, $\boldsymbol{\partial}_k$ is the vector field dual to $\mathrm{d}x^i \wedge \mathrm{d}x^j$. Hence, all of the vector calculus gibberish can be entirely worked with the more elegant language of differential forms.

Tackling the LHS of (14),

$$\begin{aligned}\int_{\mathcal{V}_2} \operatorname{curl}(\boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{\Sigma} &= \int_{\mathcal{V}_2} \operatorname{curl}(\boldsymbol{A}) \cdot \boldsymbol{n}\, \mathrm{d}\Sigma \\ &= \int_{\mathcal{V}_2} (\partial_2 A^3 - \partial_3 A^2, \partial_3 A^1 - \partial_1 A^3, \partial_1 A^2 - \partial_2 A^1) \cdot (0,0,1)\, \mathrm{d}x^1 \mathrm{d}x^2 \\ &= \int_{\mathcal{V}_2} (\partial_1 A^2 - \partial_2 A^1)\, \mathrm{d}x^1 \mathrm{d}x^2. \end{aligned} \tag{17}$$

This confirms the equality of the LHS of both (7) and (14). Straightforward calculations show the rest:

- (RHS of (7))
$$\int_{\partial \mathcal{V}_2} \tilde{\boldsymbol{A}} = \int_{\partial \mathcal{V}_2} A_i \mathrm{d}x^i.$$
- (RHS of (14))
$$\int_{\partial \mathcal{V}_2} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \int_{\partial \mathcal{V}_2} {}^{E}g_{ij} A^i \mathrm{d}x^j = \int_{\partial \mathcal{V}_2} A_j \mathrm{d}x^j,$$
  where on the second equality we used the (geometric) definition of the dot product, which is merely a contraction given by the metric tensor. Equality of the RHS of both (7) and (14) has been established.

Showing the validity of (7) as a generalization of (15) is at this point a straightforward application of everything we have discussed, so it is left to the reader.