Orthogonal Polynomials, Verblunsky Coefficients, and a Szegő–Verblunsky Theorem on the Unit Sphere in $\mathbb{C}^d$

Connor Gauntlett

Given a measure μ on the unit sphere $\partial\mathbb{B}^d$ in $\mathbb{C}^d$ with Lebesgue decomposition $\mathrm{d}\mu = w\,\mathrm{d}\sigma + \mathrm{d}\mu_s$, with respect to the rotation-invariant Lebesgue measure σ on $\partial\mathbb{B}^d$, this talk shall generalise to several variables notions of orthogonal polynomials, Verblunsky coefficients in the unit disc, and an associated Christoffel function.

We highlight a recurrence relation for the orthogonal polynomials involving the Verblunsky coefficients $(\gamma_{0,\alpha})_{\alpha\in\mathbb{N}_0^d}$ which is reminiscent of the classical Szegő recurrences, and build these ideas up to a multivariate analogue of the classical Szegő–Verblunsky theorem: namely, if $\operatorname{supp}\mu_s$ is discrete and there exists $f \in H^\infty(\mathbb{B}^d)$ with $f(0) = 1$ and

$$\int_{\partial\mathbb{B}^d} |f(\zeta)|^2 w(\zeta)\mathrm{d}\sigma(\zeta) \leq \exp\left(\int_{\partial\mathbb{B}^d} \log\left(w(\zeta)\right)\mathrm{d}\sigma(\zeta)\right),$$

then we show that

$$\prod_{\alpha\in\mathbb{N}_0^d} (1 - |\gamma_{0,\alpha}|^2) = \exp\left(\int_{\partial\mathbb{B}^d} \log\left(w(\zeta)\right)\mathrm{d}\sigma(\zeta)\right).$$

Time permitting, we shall outline two classes of measures for which these assumptions hold (i.e. for which the multivariate Szegő–Verblunsky theorem holds) and discuss an example of a weight w lying outside of these classes for which we can directly verify that $\prod_{\alpha\in\mathbb{N}_0^d}(1-|\gamma_\alpha|^2) \neq \exp\left(\int_{\partial\mathbb{B}^d} \log\left(w(\zeta)\right)\mathrm{d}\sigma(\zeta)\right)$.

This talk is based on joint work with David Kimsey.