Topological Complexity of Superfluid Turbulence

Robert G. Cooper

1. Aims

- To numerically simulate a patch of superfluid turbulence away from boundaries and to reach a statistically steady state independently of the arbitrary initial condition
- To relate the topological complexity of superfluid vortices with the geometry and dynamics of the system.

2. Simulating Turbulence

We numerically simulate a patch of superfluid turbulence, reaching a statistically steady state in the absence of solid or periodic boundaries. Superfluid vortices are modelled according to the Vortex Filament Method outlined in [1] at $T = 1.9$K.

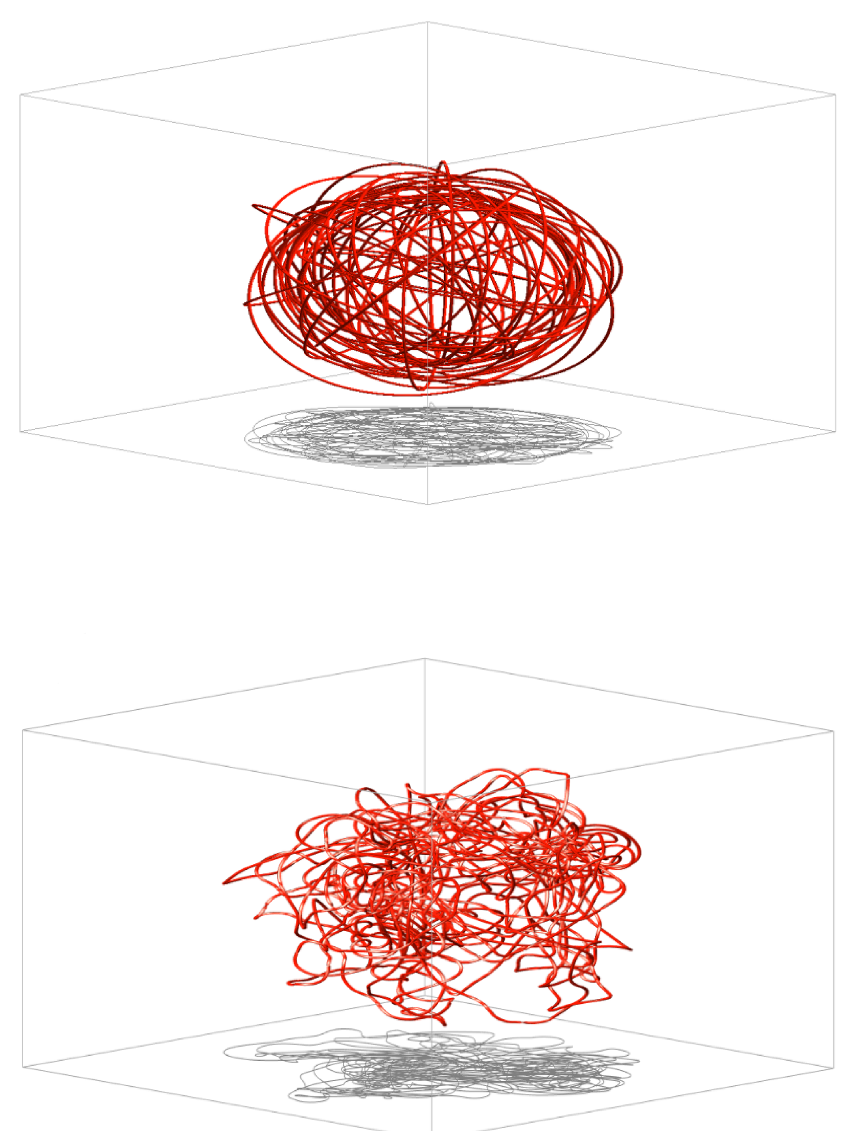
Turbulence is driven by the normal fluid velocity, $\mathbf{v}_n(r,t) = \mathbf{v}_{ABC}(r,t)exp(-r/d)$ where $\mathbf{v}_{ABC}$ is the Arnold-Beltrami-Childress (ABC) flow defined by

$$\mathbf{v}_A = \begin{pmatrix} Bcos(ky - \omega t) + Csin(kz - \omega t) \\ Ccos(kz - \omega t) + Asin(kx - \omega t) \\ Acos(kx - \omega t) + Bsin(ky - \omega t) \end{pmatrix}$$

where we set $A = B = C$, $k = 8\pi/d$ with $d = 1.5$cm, ω is the normal fluid frequency which we set to 1s^{-1} and $r = \sqrt{x^2 + y^2 + z^2}$.

3. Results

We observe that the initial vortex configuration expands in length due to instability with the tightly packed vortices reconnecting and releasing Kelvin waves. The vortex lines decay as they leave the central region due to friction with the normal fluid which is stationary away from the centre. The overall topology of the system is continuously altered by reconnections and the tangle quickly reaches a statistically steady turbulent state. An example of the tangle evolution is seen in Figure 1 with a drive of $A = 1$.



(a).

(b).

Figure 1: Illustrations of the typical time evolution of the vortex tangle with a drive of $A = 1$. The vortex lines are the red curves, equally scaled and enclosed in a box with shadows for visualisation purposes only. The figure shows the vortex tangle at (a): The initial condition, $t = 0.00s$ consisting of 40 randomly oriented loops with sizes varying according to a normal distribution with an average number of 200 points, located at the centre of the region. (b): $t = 0.1.96s$, once the superfluid has reached a statistically steady state.

4. Topological Complexity

To describe the topology of each vortex loop we compute a knot invariant known as the Alexander polynomial, $\Delta(\tau)$ and quantify topological complexity using the order ν of the polynomial as in [2]. So for example, the trivial *unknot/circle* has $\Delta(\tau) = 1$ with $\nu = 0$ and a *trefoil* has $\Delta(\tau) = 1 - \tau + \tau^2$ of order $\nu = 2$. A vortex loop, j with Alexander polynomial

$$\Delta_j(\tau) = a_0 + a_1\tau + ... + a_{\nu_j}\tau^{\nu_j}$$

is of order ν_j.

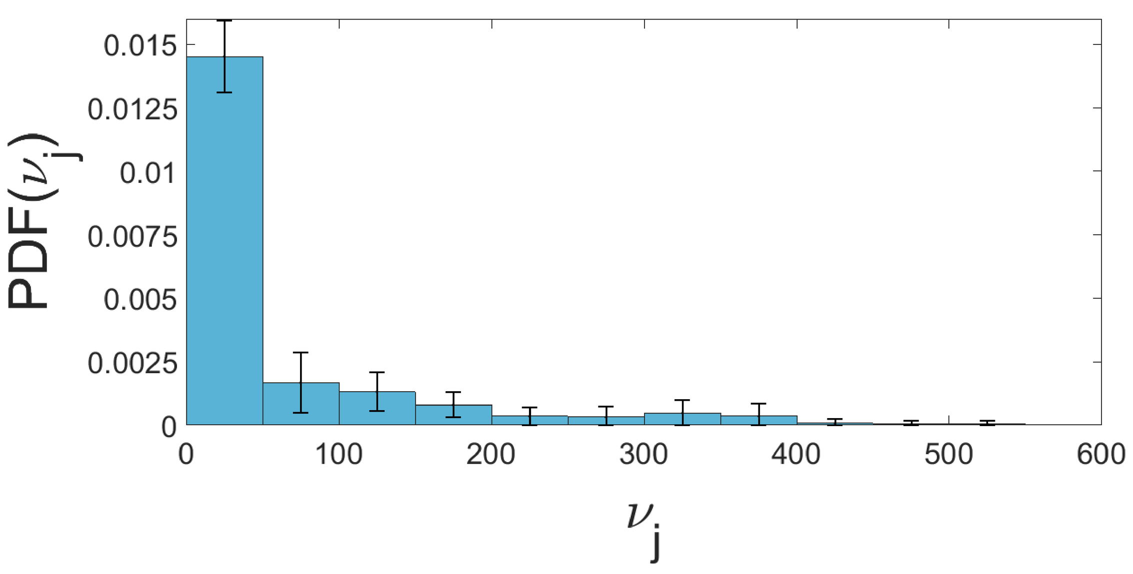


Figure 2: For all drives, and at all time steps we see that there are many vortex loops of low topological complexity but that there always exist highly complex vortices with large Alexander polynomials.

We see in Figure 2 that we encounter **many** vortex loops of **low complexity** but that there always exist **one or two** loops of **high complexity** in the system at each time step over all drives.

The writhe is a property of knots which sums numerical values assigned to each apparent crossing, giving an overall measure of the geometry of the configuration. We will thus compare the topological complexity with geometry using the lengths of vortices, Λ_j and their writhe, Wr_j. Figure 3 shows that ν_j **increases with vortex length** Λ_j and in Figure 4, ν_j appears to **increase linearly with the writhe**, Wr_j.

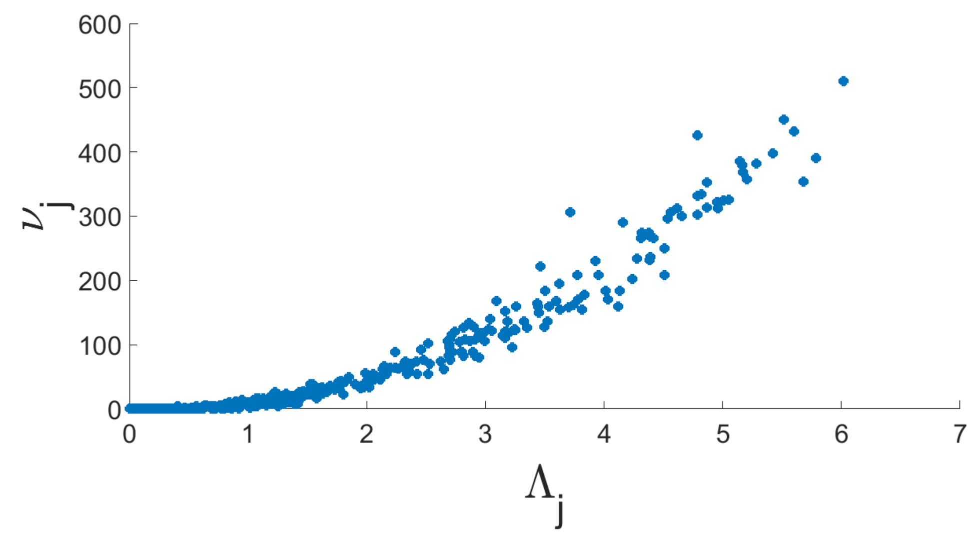


Figure 3: The order of the Alexander polynomial, ν_j for each vortex against vortex length, Λ_j in cm.

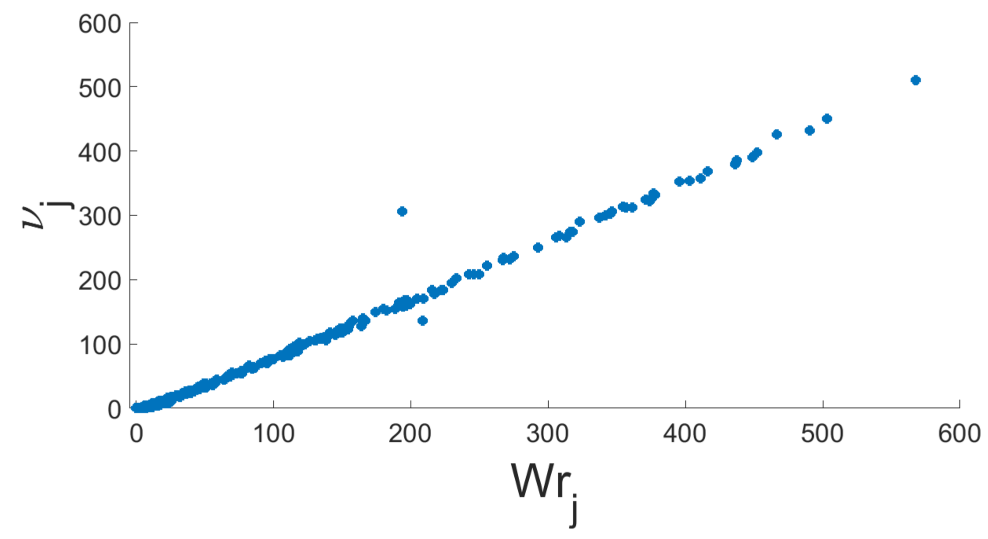


Figure 4: Comparing the order of the Alexander polynomial, ν_j with the writhe, Wr_j.

When comparing the order of the Alexander polynomial, ν_j with the drive parameter, A of the normal fluid in Figure 5 we see that in general, **topological complexity increases with drive**.

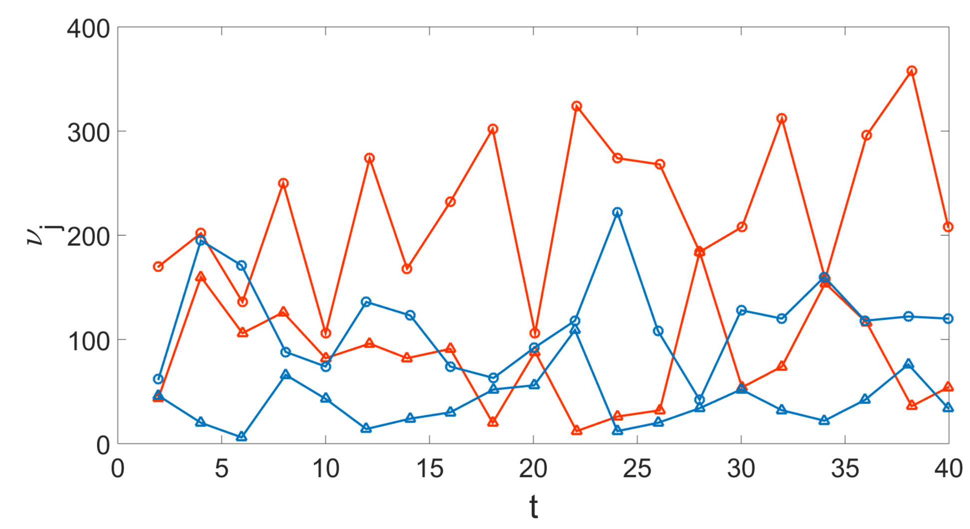


Figure 5: The time evolution of the highest and second highest ordered Alexander polynomials, ν_1 (circles) and ν_2 (triangles) respectively for a higher drive of $A = 1.1$ (red) and lower drive of $A = 1.0$ (blue).

5. Conclusions

- At all time steps and for all drives, we see that the turbulence consists of many vortices of low complexity and at least one of high complexity.
- Topological complexity increases with vortex length and increases linearly with the writhe.
- In general, driving the system more intensely results in higher topological complexity.

References

[1] Saffman P.G. (1992), *Vortex Dynamics*, (Cambridge University Press)

[2] Mezgarnezhad M., Cooper R.G., Baggaley A.W. and Barenghi C.F. (2016), Helicity and Topology of a Small Region of Quantum Vorticity, *arXiv:1610.10024* (*To be published in Fluid Dyn. Res. in 2017*)